

Effective theory of color superconductivity

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We briefly review an effective theory of QCD at high baryon density, describing the relevant modes near the Fermi surface. The high density effective theory has properties of reparametrization invariance and gauge invariance, maintained in a subtle way. It also has a positive measure, allowing lattice simulations at high baryon density. We then apply it to gapless superconductors and discuss recent proposals to resolve the magnetic instability of gapless superconductivity.

§1. Introduction

Quantum Chromodynamics (QCD) is now the accepted theory of strong interactions, consistent with all experimental data. One of the salient features of QCD is that strong interactions become weaker and weaker at high energy or at short distances due to the chromomagnetic interaction of gluons, a phenomenon called “asymptotic freedom”. The prediction of QCD on how the coupling depends on the scale has been well tested by numerous experiments. The logical consequence of the asymptotic freedom is then that hadronic matter, bound by the strong interactions, must undergo a phase transition to quark matter, when the hadronic matter gets extremely squeezed or heated up, at the critical temperature or the critical chemical potential being of order of Λ_{QCD} , the characteristic scale of QCD.

The phase transition of QCD at finite temperature but at zero density has been confirmed by lattice calculations to occur at temperature about 200 MeV.¹⁾ The lattice calculation of QCD at finite baryon density, however, has not made much progress, since it suffers from the notorious sign problem. The Euclidean partition function of QCD at finite density, used for lattice simulations, does not have a positive-definite measure, because the determinant of the Dirac operator, $M = \gamma_E^\mu D_E^\mu + \mu \gamma_E^4$, is no longer positive definite, when the chemical potential, μ , is present.

There have been several attempts to overcome this difficulty at finite but small density by several methods.^{2),3)} At high density it was recently pointed out that the sign problem is due to the fast modes, quarks with energy much larger than the chemical potential, suggesting that the sign problem is either mild or absent for physics near the Fermi surface.⁴⁾

While the lattice simulation struggles to overcome the sign problem, there have been much progress in understanding QCD at high baryon density (or dense QCD for short) from the theoretical side,⁵⁾ using effective theories or models of dense QCD. The ground state of dense QCD is believed to be a color superconducting state, where quarks near the Fermi surface form Cooper pairs. Unlike the electron

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superconductor, where the pairing is mediated by phonons, the Cooper-pairing in quark matter is due to the gluon exchange interaction, since quarks in antisymmetric in color are attractive to each other. Though the quark-antiquark channel is most attractive, it is costly to form quark-antiquark condensate since quarks in Dirac sea has to be excited above the Fermi sea.

Recently, it is found that the color superconducting quark matter exhibits rather rich phases because quarks have several flavors with different masses, interacting electroweakly.⁶⁾ One interesting phase is the so-called gapless color superconducting state,⁷⁾ where the Fermi sea is partially filled, though quarks do pair, allowing gapless excitation of quarks. However, it is soon pointed out that the gapless color superconductor has imaginary Meissner mass, indicating its instability.⁸⁾ In this talk we discuss the high density effective theory of dense QCD and apply it to study the gapless color superconductivity and propose two resolutions of the instability of the gapless superconductor.^{9),10)}

§2. High Density Effective Theory

The relevant degrees of freedom of dense QCD at low energy are quarks and holes near the Fermi surface together with screened gluons, since quarks deep in the Fermi sea and also antiquarks are decoupled from the low-energy dynamics due to Pauli blocking. The effective theory of quarks and holes in dense QCD is derived in references.^{11),12)}

Since the quark chemical potential in dense QCD is very large, $\mu \gg \Lambda_{\text{QCD}}$, the velocity of quarks and holes is conserved under a typical QCD interaction and thus we may decompose the momentum of quarks near the Fermi surface

$$p^\mu = \mu v^\mu + l^\mu, \quad (2.1)$$

where the residual momentum $|l^\mu| < \mu$ and $v^\mu = (0, \vec{v}_F)$ with Fermi velocity \vec{v}_F . The effective Lagrangian of quarks and holes is given as

$$\mathcal{L}_Q = Z_{||} \bar{\psi}_+ i \gamma_{||} \cdot D \psi_+ - Z_{\perp} \bar{\psi}_+ \gamma^0 \frac{(\gamma_{\perp} \cdot D)^2}{2\mu} \psi_+ + \dots, \quad (2.2)$$

where $D = \partial - ig_s A$ is the covariant derivative and $\gamma_{||}^\mu = (\gamma^0, \hat{v}_F \hat{v}_F \cdot \vec{\gamma}) = \gamma^\mu - \gamma_{\perp}^\mu$,

$$\psi_+(\vec{v}_F, x) = \frac{1 + \vec{\alpha} \cdot \hat{v}_F}{2} e^{-i\mu \vec{v}_F \cdot \vec{x}} \psi(x), \quad (2.3)$$

with $\vec{\alpha} = \gamma^0 \vec{\gamma}$ and $\hat{v}_F = \vec{v}_F / |\vec{v}_F|$. We note that the effective theory enjoys a reparametrization invariance,

$$\vec{v}_F \mapsto \vec{v}_F + \frac{\delta \vec{l}_{\perp}}{\mu}, \quad \vec{l} \mapsto \vec{l} - \delta \vec{l}, \quad (2.4)$$

under which the Lagrangian (2.2) is invariant. The reparametrization invariance is quite useful when we calculate the loop corrections, since it restricts the form of counter terms. For instance the reparametrization invariance requires $Z_{||} = Z_{\perp}$.

The Lagrangian for gluons contains in addition to the usual kinetic term the terms coming from the modes far away from the Fermi surface,

$$\mathcal{L}_G = -\frac{1}{4g_s^2}F_{\mu\nu}^2 - \frac{1}{2}M^2 A_\mu A_\nu g_\perp^{\mu\nu} + \dots, \quad (2.5)$$

where the Debye screening mass $M^2 = N_f g_s^2 \mu^2 / (2\pi^2)$ for N_f flavors. ^{*)} The high density effective theory of dense QCD is then given as

$$\mathcal{L}_{\text{HDET}} = \mathcal{L}_Q + \mathcal{L}_G. \quad (2.6)$$

It is important that one should include the Debye screening mass and also higher order terms in (2.5) to maintain the gauge invariance, since the Fermi surface is not gauge invariant and ψ_+ modes alone are not enough to maintain the gauge-invariance. Under a gauge transformation, $U(x) = e^{i\vec{q}\cdot\vec{x}}$, the energy level shifts and so does the Fermi surface (see Fig. 1)

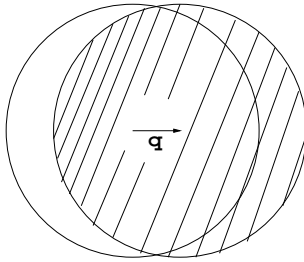


Fig. 1. Spectral Flow. The circle denotes a Fermi surface.

$$E = \vec{l} \cdot \vec{v}_F \mapsto E = \vec{l} \cdot \vec{v}_F + \vec{q} \cdot \vec{v}_F. \quad (2.7)$$

Since the mode decomposition of quark fields into modes near the Fermi surface and modes far away from the Fermi surface is not gauge invariant, the effect of modes away from the Fermi surface should be added in the effective Lagrangian otherwise the gauge-invariance will be lost. Let's consider for instance the current-current correlation function,

$$\Pi^{\mu\nu}(p) = \int_x e^{ip \cdot x} \langle J^\mu(x) J^\nu(0) \rangle = -iM^2 \int \frac{d\Omega_{\vec{v}_F}}{4\pi} \left(\frac{-2\vec{p} \cdot \vec{v}_F V^\mu V^\nu}{p \cdot V + i\epsilon \vec{p} \cdot \vec{v}_F} + g_\perp^{\mu\nu} \right), \quad (2.8)$$

where $\epsilon \rightarrow 0^+$. We see that the Debye mass term or $g_\perp^{\mu\nu}$ in the parenthesis in Eq. (2.8) is essential for the current conservation and thus for the gauge invariance,

$$p_\mu \Pi^{\mu\nu}(p) = 0. \quad (2.9)$$

^{*)} The ellipsis denotes terms with higher powers of gauge fields, which are known as hard-dense-loop Lagrangian.¹³⁾

§3. Positivity at asymptotic density

For quarks very near the Fermi surface ($E \ll \mu$) the Fermi surface is almost flat and one can show that there is a symmetry that pairs the eigenvalues of the Dirac operator:

$$M_{\text{eff}} = \gamma_{\parallel}^E \cdot D(A) = \gamma_5 M_{\text{eff}}^{\dagger} \gamma_5. \quad (3.1)$$

Since the eigenvalues of the Dirac operator is pure imaginary, $i\lambda$ (λ being real), the determinant of the Dirac operator is positive semi-definite:

$$\det M_{\text{eff}} = \det M_{\text{eff}}^{\dagger} = \left(\prod_{\lambda} \lambda^2 \right)^{1/2}. \quad (3.2)$$

After integrating out the fast modes, the partition function for dense QCD becomes

$$Z(\mu) = \int dA \det M_{\text{eff}}(A) e^{-S_{\text{eff}}(A)}, \quad (3.3)$$

and the effective action is given as

$$S_{\text{eff}}(A) = \int d^4x_E \left(\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{M^2}{16\pi} \sum_{\vec{v}_F} A_{\perp\mu}^a A_{\perp\mu}^a \right) + \cdots, \quad (3.4)$$

where the ellipsis denotes terms with higher powers of the gauge fields and also the terms coming from the higher order terms in the quark Lagrangian (2.2).

We see that the measure of the partition function is positive definite if we do not keep the higher order terms in the effective action, allowing us to simulate on a lattice. One byproduct of the positivity at asymptotic density is that one can use the Vafa-Witten theorem¹⁴⁾ to prove the Color-Flavor-Locked phase¹⁵⁾ is the true ground state at very high baryon density,⁴⁾ since the vector symmetry can not be spontaneously broken when the measure is positive.

The size of the higher order terms can be estimated, using the naive dimensional analysis. We find the size of corrections to the leading terms to be at a scale $\Lambda \ll \mu$ $\frac{\alpha_s}{2\pi} \frac{\Lambda}{\mu}$. Therefore the sign problem of dense QCD is mild or absent if $\alpha_s \Lambda \ll 2\pi\mu$.

§4. Application: New phases in gapless superfluids

The color superconductivity has a rich phase structure due to stress on pairing quarks. Since quarks in dense matter pair with different flavors for spin-zero pairing, the flavor-dependent mass and the electroweak interaction among different flavors leads to the difference in the chemical potential, $2\delta\mu$, between pairing quarks.

If we solve the Cooper-pair gap equation for dense matter under stress,

$$\frac{\partial \Omega}{\partial \Delta} = 0, \quad (4.1)$$

we find two branches of solutions (See Fig. 2): When $\Delta > \delta\mu$, $\Delta = \Delta_0$, the BCS phase, and when $\Delta < \delta\mu$, $\Delta = \sqrt{\Delta_0(2\delta\mu - \Delta_0)}$, the Sarma phase.¹⁶⁾ The BCS

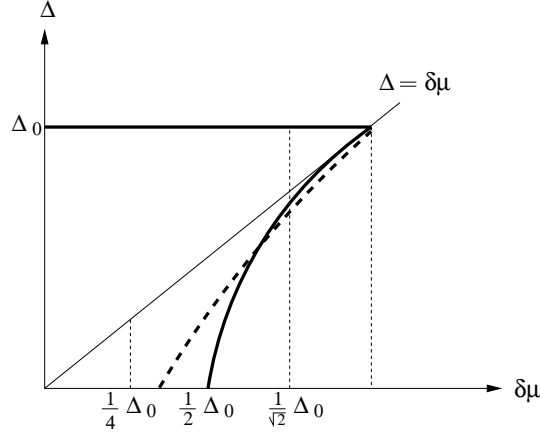


Fig. 2. Solutions to the gap equation and the charge neutrality equation.

phase is unstable for $\delta\mu > \Delta_0/\sqrt{2}$, known as the Clogston instability:¹⁷⁾ When the stress is big enough, the Cooper pairs break. On the other hand, the Sarma phase is unstable for all values of $\delta\mu$ in $\frac{1}{2}\Delta_0 < \delta\mu < \Delta_0$, for which the solution exists.

Recently it was pointed out that the Sarma phase might be stable if one imposes the charge neutrality condition on quark matter,⁷⁾

$$\frac{\partial\Omega}{\partial\mu_Q} = 0, \quad (4.2)$$

where μ_Q is the electric charge chemical potential. (The charge neutrality has to be satisfied for quark matter inside of compact stars, otherwise too much energy will build up at the surface of quark matter, destabilizing whole system.) The free energy difference between the neutral Sarma phase and the neutral quark matter is found to be in the mean field approximation¹⁸⁾

$$\delta\Omega|_{Q=0} = \Omega_{\text{Sarma}}|_{Q=0} - \Omega_{\text{free}}|_{Q=0} = -\frac{\bar{\mu}^2}{2\pi^2} (\delta\mu_0)^2 \left[1 - \left(\frac{\Delta_0}{2\delta\mu_0} - 1 \right)^2 \right], \quad (4.3)$$

where $\bar{\mu}$ is the average chemical potential and $\delta\mu_0$ is half of the chemical potential difference of neutral, unpaired quark matter. The charge neutrality condition stabilizes the Sarma phase if

$$\frac{\Delta_0}{4} \leq \delta\mu_0 \leq \frac{\Delta_0}{\sqrt{2}}. \quad (4.4)$$

The energy dispersion relation of quarks in the Sarma phase is given as

$$\omega(\vec{p}) = \pm \left(\delta\mu \pm \sqrt{\epsilon^2(\vec{p}) + \Delta^2} \right), \quad (4.5)$$

which is plotted in Fig. 3. At low energy the modes near the Fermi surface are relevant. The modes near the Fermi surface are gapless and have either quadratic

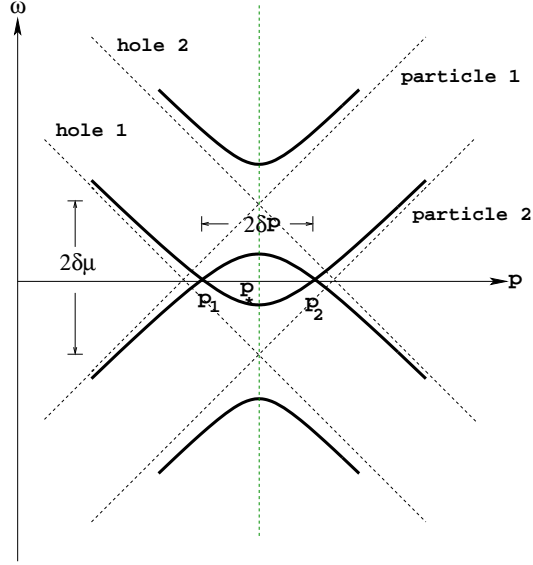


Fig. 3. The energy dispersion relation of quasi quarks and holes. The Fermi sea is partially filled, having inner and outer Fermi surfaces at p_1 and p_2 .

or linear dispersion relations, depending on the strength of the stress:

$$\omega(\vec{p}) \simeq \begin{cases} \eta \vec{v}_i \cdot \vec{l}, & \text{if } |\vec{v}_i \cdot \vec{l}| < \delta p \\ \frac{(\vec{v}_i \cdot \vec{l})^2}{2\delta\mu}, & \text{if } \delta p < |\vec{v}_i \cdot \vec{l}| < \delta\mu \\ \vec{v}_i \cdot \vec{l}, & \text{if } \bar{\mu} \gg |\vec{v}_i \cdot \vec{l}| > \delta\mu, \end{cases} \quad (4.6)$$

where \vec{v}_1 and \vec{v}_2 are the Fermi velocities of quarks near the Fermi surfaces, p_1 and p_2 , respectively.

When $\delta\mu \approx \Delta$ or $\delta p \approx 0$, the gapless modes are quadratic all the way down to the Fermi surface at $p_* \simeq p_1 \simeq p_2$. After integrating out the irrelevant degrees of freedom, we get an effective Lagrangian for quadratic gapless modes,

$$\mathcal{L}_{\text{eff}} = \sum_{\vec{v}_*} \Psi^\dagger(\vec{v}_*, x) \left[i\partial_t + \frac{(\vec{v}_* \cdot \vec{\nabla})^2}{2\delta\mu} \right] \Psi(\vec{v}_*, x) + \frac{\kappa}{2} (\Psi^\dagger \Psi)^2 + \dots \quad (4.7)$$

Under the scaling $E \mapsto s E$, the action $S = \int dt d^3l \mathcal{L}_{\text{eff}}$ is invariant and we have

$$\vec{l}_\parallel \mapsto s^{1/2} \vec{l}_\parallel, \quad \psi_{\vec{v}_*}(t, \vec{l}) \mapsto s^{-1/4} \psi_{\vec{v}_*}(t, \vec{l}). \quad (4.8)$$

The four Fermi-interaction is relevant for incoming fermions with opposite momenta ^{*)}, regardless of the sign,

$$\kappa \mapsto s^{-1/2} \kappa. \quad (4.9)$$

^{*)} If not opposite, the interaction is marginal.

The four-Fermi interaction generated by the quantum effects of the irrelevant modes will open a secondary gap at the Fermi surface, which is only power-suppressed in couplings,^{9),19)}

$$\Delta_s \simeq 6.85 \kappa^2 \left(\frac{\nu_*}{v_*} \right)^2 \delta\mu. \quad (4.10)$$

When the stress is large enough or the infrared cutoff for the quadratic modes

$$\Lambda_{\text{quadratic}}^{\text{IR}} = \frac{1}{2\delta\mu} (\delta\mu^2 - \Delta^2) > \Delta_s, \quad (4.11)$$

the secondary gap does not open, because the RG running is not long enough. The system is then dominated by linearly gapless modes at $\omega < \Lambda_{\text{quadratic}}^{\text{IR}}$.

Integrating out all the modes except the linearly gapless modes, we get an effective potential for the gauge fields with $V_1^\mu = (\eta\bar{q} + \delta q, \eta\bar{q} \vec{v}_1)$ and $V_2^\mu = (\eta\bar{q} - \delta q, \eta\bar{q} \vec{v}_2)$,

$$V(A) = \frac{1}{2} m_M^2 \vec{A}^2 - \frac{\eta}{3} (\nu_1 v_1^2 + \nu_2 v_2^2) e^2 \bar{q}^2 \vec{A}^2 \left[\ln \left(\frac{M^2}{\vec{A}^2} \right) + 3 \right] \quad (4.12)$$

$$- \frac{1}{2} m_D^2 A_0^2 - \frac{1}{\eta} (\nu_1 + \nu_2) (\eta\bar{q} + \delta q)^2 e^2 A_0^2 \left[\ln \left(\frac{M^2}{A_0^2} \right) + 3 \right], \quad (4.13)$$

where ν_1 and ν_2 are the area of the inner and outer Fermi surfaces, respectively, and the electric charges of pairing quarks are $q_1 = \bar{q} + \delta q$ and $q_2 = \bar{q} - \delta q$, upon imposing the renormalization conditions,

$$\left. \frac{\partial^2 V}{\partial A_0^2} \right|_{A_0=M} = -m_D^2, \quad \frac{1}{3} \delta_{ij} \left. \frac{\partial^2 V}{\partial A_i \partial A_j} \right|_{\vec{A}^2=M^2} = m_M^2. \quad (4.14)$$

The minimum of the potential occurs at

$$\left\langle \left(\vec{A} - \vec{\nabla} \varphi \right)^2 \right\rangle \simeq \delta\mu^2 \exp \left[2 - \frac{4\nu_* v_*^2}{\eta (\nu_1 v_1^2 + \nu_2 v_2^2)} \right]. \quad (4.15)$$

We find that gapless superfluids are stabilized by spontaneously generating Nambu-Goldstone currents. One can further show that the Meissner mass for this phase is nonnegative but directional.⁹⁾

§5. Conclusions

We have discussed an effective theory of dense QCD, called high density effective theory. The theory has a reparametrization invariance and the gauge invariance is maintained by counter terms. Furthermore the theory has a positive semi-definite measure at the leading order. Lattice simulation should be possible for high density quark matter. By the positivity at high density one can establish a rigorous theorem like the Vafa-Witten theorem.

As an application of the effective theory, we study the gapless color superconductivity, which occurs when the color superconductors are under stress. When the

stress is comparable to the gap, the gapless modes have a quadratic dispersion relation and open a secondary gap at the Fermi Surface, stabilizing the system. The secondary gap is only power suppressed in couplings due to peculiar scaling properties of the quadratic gapless modes. When the stress is much larger than the gap, the relevant modes are linearly gapless modes. By calculating the Coleman-Weinberg potential for the gauge fields, we show that the gapless superfluids become stabilized by spontaneously generating Nambu-Goldstone currents.

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